POLYNOMIAL DETECTION OF MATRIX SUBALGEBRAS

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Abstract. The double Capelli polynomial of total degree $2t$ is

$$\sum \left\{ (\operatorname{sg} \sigma\tau) x_{\sigma(1)} y_{\tau(1)} x_{\sigma(2)} y_{\tau(2)} \cdots x_{\sigma(t)} y_{\tau(t)} \middle| \, \sigma, \, \tau \in S_t \right\}.$$

It was proved by Giambruno-Sehgal and Chang that the double Capelli polynomial of total degree $4n$ is a polynomial identity for $M_n(F)$. (Here, F is a field and $M_n(F)$ is the algebra of $n \times n$ matrices over F). Using a strengthened version of this result obtained by Domokos, we show that the double Capelli polynomial of total degree $4n-2$ is a polynomial identity for any proper F-subalgebra of $M_n(F)$. Subsequently, we present a similar result for nonsplit extensions of full matrix algebras.

1. Introduction

The double Capelli polynomial of total degree $2t$ is

$$\sum \left\{ (\operatorname{sg} \sigma\tau) x_{\sigma(1)} y_{\tau(1)} x_{\sigma(2)} y_{\tau(2)} \cdots x_{\sigma(t)} y_{\tau(t)} \middle| \, \sigma, \, \tau \in S_t \right\}.$$

In this paper we show that the double Capelli polynomial of degree $4n-2$ is a polynomial identity for any proper subalgebra of $M_n(F)$. Subsequently, we present a polynomial test for nonsplit extensions of full matrix algebras.

To begin, let F be a field, $M_n(F)$ the algebra of $n \times n$ matrices over F, and $F\{X\} = F\{x_1, x_2, \dots\}$ the free associative algebra over F in countably many variables. Sometimes we will use other variables x, y, z, x_i, y_i for notation simplicity. A nonzero polynomial $f(x_1, \dots x_m) \in F\{X\}$ is a *polynomial identity* for an F-algebra R if $f(r_1, \dots, r_m) = 0$ for all $r_1, \dots, r_m \in R$. A *T-ideal* is an ideal of $F\{X\}$ which is closed under endomorphisms of $F\{X\}$. If $f_1, \dots, f_t$ are polynomial identities for R, so is every polynomial f in the T-ideal generated by $f_1, \dots, f_t$. In this case we say that the identity $f = 0$ in R is a *consequence* of the identities $f_i = 0$, for $1 \leq i \leq t$.

The standard polynomial of degree t is

$$s_t(x_1, \dots, x_t) = \sum_{\sigma \in S_t} (\operatorname{sg}\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(t)},$$

where S_t is the symmetric group on $\{1, \dots, t\}$ and $(\operatorname{sg}\sigma)$ is the sign of the permutation $\sigma \in S_t$. The standard polynomial s_t is homogeneous of degree t, multilinear and alternating.

The Amitsur-Levitzki theorem asserts that $M_n(F)$ satisfies any standard polynomial of degree $2n$ or higher. Moreover, if $M_n(F)$ satisfies a polynomial of degree

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$2n$, then the polynomial is a scalar multiple of s_{2n} (cf. [1]). The *Capelli polynomials* are

$$c_{2t-1}\left(x_1,\ldots,x_t,y_1,\ldots,y_{t-1}\right)=\sum_{\sigma\in S_t}(\mathrm{sg}\sigma)x_{\sigma(1)}y_1x_{\sigma(2)}y_2\cdots x_{\sigma(t-1)}y_{t-1}x_{\sigma(t)},$$

and

$$c_{2t}\left(x_1,\ldots,x_t,y_1,\ldots,y_t\right)=c_{2t-1}\left(x_1,\ldots,x_t,y_1,\ldots,y_{t-1}\right)y_t.$$

These polynomials were introduced by Razmyslov in [9]. The polynomials c_{2t-1} and c_{2t} are multilinear and alternating as a function of $x_1,\ldots,x_t$. It is clear by a dimension argument that c_{2n^2} is a PI for any proper F-subalgebra of $M_n(F)$. On the other hand, c_{2n^2} is not a PI for $M_n(F)$. To see this, evaluate $c_{2n^2}\left(x_1,\ldots,x_{n^2},y_1,\ldots y_{n^2}\right)$ with

$$\left(x_1,x_2,\ldots,x_n,x_{n+1},\ldots x_{n^2-1},x_{n^2}\right)=\left(e_{11},e_{12},\ldots,e_{1n},e_{21},\ldots e_{n(n-1)},e_{nn}\right),$$
$$\left(y_1,\ldots,y_n,\ldots y_{n^2-1},y_{n^2}\right)=\left(e_{11},\ldots,e_{n2},\ldots e_{(n-1)n},e_{n1}\right).$$

where the e_{ij} are the standard matrix units, $y_1=e_{11}$, $y_{n^2}=e_{n1}$, and $y_2,\ldots y_{n^2-1}$ are the unique choices of matrix units such that the monomial with $\sigma=1$ is nonzero, so c_{2n^2} takes on the value $e_{11}\neq 0$. Based on this example, we introduce the following definition:

Definition 1.1. We will say that a multilinear polynomial $f(x_1,\ldots,x_t)\in F\{X\}$ is a *polynomial test* for an F-algebra R if it is not a polynomial identity for R but it is an identity for every proper F-subalgebra of R.

Thus, the Capelli polynomial of total degree $2n^2$ is a polynomial test for $M_n(F)$. Moreover, central polynomials for $M_n(F)$ are polynomial tests for $M_n(F)$ (see [6]). In [2], it is proved that the standard polynomial of degree $2n-2$ is a polynomial test for the subalgebra of upper triangular matrices of $M_n(F)$. The *double Capelli polynomials* are

$$h_{2t-1}(x_1,\ldots,x_t,y_1,\ldots,y_{t-1})$$
$$=\sum_{\sigma\in S_t,\tau\in S_{t-1}}(\mathrm{sg}\sigma\tau)x_{\sigma(1)}y_{\tau(1)}x_{\sigma(2)}y_{\tau(2)}\cdots x_{\sigma(t-1)}y_{\tau(t-1)}x_{\sigma(t)},$$

and

$$h_{2t}(x_1,\ldots,x_t,y_1,\ldots,y_t)$$
$$=\sum_{\sigma,\tau\in S_t}(\mathrm{sg}\sigma\tau)x_{\sigma(1)}y_{\tau(1)}x_{\sigma(2)}y_{\tau(2)}\cdots x_{\sigma(t-1)}y_{\tau(t-1)}x_{\sigma(t)}y_{\tau(t)}.$$

Note that h_{2t-1} and h_{2t} are multilinear and alternate in the x_i and also in the y_j.

Formanek pointed out that h_{4n-2} is not a polynomial identity for $M_n(F)$ and asked for the least integer m such that h_m is a polynomial identity for $M_n(F)$. Chang [3] has proved that the double Capelli polynomial h_{2t} is a consequence of the standard polynomial s_t. A different proof that $h_{4n}=0$ is a polynomial identity for $M_n(F)$, that uses a variation of Rosset's proof of the Amitsur-Levitzki theorem [10], was given by Giambruno-Sehgal in [7]. An elegant one-line proof of Domokos is given in [4], Example 2.2, p. 917.

In [5], Domokos obtained a generalization of Chang's theorem. Since it is important in these notes, the precise statement of Domokos's theorem is included below.

Let $x_1, \dots, x_d, y_1, \dots, y_m$ be noncommuting variables over F, and let $w_1, \dots, w_u$ be monomials in $y_1, \dots, y_m$ such that $w_1, \dots, w_u$ is a reordering of $y_1, \dots, y_m$. For a subset $\Pi \subseteq S_d$ and a monomial partition $\{w_1, \dots, w_u\}$ of the set of variables Y we put

$$\begin{aligned} f_\Pi(x_1, \dots, x_d, y_1, \dots, y_m | w_1, \dots, w_u) = \\ \sum (\operatorname{sg} \mu) x_{\pi(1)} \cdots x_{\pi(d_1)} w_{\rho(1)} x_{\pi(d_1+1)} \cdots x_{\pi(d_1+d_2)} w_{\rho(2)} \cdots \\ \cdots w_{\rho(s)} x_{\pi(d_1+\cdots+d_s+1)} \cdots x_{\pi(d_1+\cdots+d_{s+1})}, \end{aligned}$$

where the summation runs over all $\pi \in \Pi, \rho \in S_u$, $d_i \geq 1$ for $i = 1, \dots, s+1$ such that $d_1 + \cdots d_{s+1} = d$ and $\operatorname{sg} \mu$ is ± 1 according to the parity of the permutation of the "underlying" variables $x_1, \dots, x_d, y_1, \dots, y_m$ in the corresponding term.

Theorem 1.2. [5] *The polynomial $f_{S_d}(x_1, \dots, x_d, y_1, \dots, y_m | w_1, \dots, w_u)$ is contained in the T-ideal generated by the standard polynomial s_d.*

Corollary 1.3. [5] We have the strengthened version of the result of [3] and [7] we mentioned above:

$$\sum_{\sigma \in S_{2n}, \tau \in S_{2n-1}} (\operatorname{sg} \sigma\tau) x_{\sigma(1)} y_{\tau(1)} \cdots y_{\tau(2n-1)} x_{\sigma(2n)} = 0,$$

is a polynomial identity for $M_n(F)$, moreover, it is a consequence of the standard identity $s_{2n} = 0$.

To see that h_{4n-2} is not a polynomial identity for $M_n(F)$, consider the substitution (double staircase)

$$\begin{aligned} x_1 = e_{11}, y_1 = e_{12}, x_2 = e_{22}, y_2 = e_{23}, \dots, x_n = e_{nn} \\ y_n = e_{nn}, x_{n+1} = e_{n(n-1)}, y_{n+1} = e_{(n-1)(n-1)}, \dots, x_{2n-1} = e_{21}, y_{2n-1} = e_{11} \end{aligned}$$

where the e_{ij} are the standard matrix units. The only nonzero monomials in $h_{4n-2}(x_i, y_i)$ are the $2n-1$ even cyclic permutations of $x_1 y_1 \dots x_{2n-1} y_{2n-1}$, and they all have positive sign. Thus

$$h_{4n-2}(x_1, \dots, x_{2n-1}, y_1, \dots, y_{2n-1}) = 2I - e_{11}.$$

We finish this section with two useful properties of the double Capelli polynomials.

Proposition 1.4. (a) h_{q+r} is a linear combination, with coefficients being 1 or -1 of evaluations of $h_q \, h_r$.
(b) The polynomial h_t is a consequence of the identity h_s for any $t \geq s$.

Proof. To prove (a) we show an explicit formula, where for simplicity we consider the following statement: $h_{2(q+r)-2}$ is a linear combination with coefficients being 1 or -1 of evaluations of $h_{2q-1} \, h_{2r-1}$. Let $t = q + r - 1$ We partition the set of permutations S_t by defining the equivalence relation $\sigma_1 \sim_q \sigma_2$ if the images of the interval $[1, q]$ under σ_1 and σ_2 are the same set. Similarly, We partition the set of permutations S_t by defining the equivalence relation $\tau_1 \sim_r \tau_2$ if the images of the

interval $[1, q-1]$ under τ_1 and τ_2 are the same set. Then we have

$$h_{2t}(x_1, \ldots, x_t, y_1, \ldots, y_t) = \sum_{\substack{\bar\sigma \in S_t/\sim_q \\ \bar\tau \in S_t/\sim_r}} (\mathrm{sg}\sigma\tau)\, h_{2q-1}(x_{\sigma(1)}, \ldots, x_{\sigma(q)}, y_{\tau(1)}, \ldots, y_{\tau(q-1)})$$
$$h_{2r-1}(y_{\tau(1)}, \ldots, y_{\tau(t)}, x_{\sigma(q+1)}, \ldots, x_{\sigma(t)}).$$

The assertion in (b) follows immediately from (a). $\square$

2. A polynomial test for the full matrix algebra

The main goal of this section is to prove that h_{4n-2} is a polynomial test for $M_n(F)$. Before proceeding to the proof of this theorem we need some preliminaries and notation (cf. [8]). Let ℓ, m be positive integers such that $\ell + m = n$ and set

$$E_{(\ell,m)}(F) = \begin{bmatrix} M_\ell(F) & M_{\ell\times m}(F) \\ 0 & M_m(F) \end{bmatrix},$$

an F-subalgebra of $M_n(F)$.

(i) Associated to $E_{(\ell,m)}(F)$ are canonical F-algebra homomorphisms

$$\pi_\ell \colon E_{(\ell,m)}(F) \to M_\ell(F) \quad \text{and} \quad \pi_m \colon E_{(\ell,m)}(F) \to M_m(F).$$

Further identify $M_\ell(F)$ and $M_m(F)$ with

$$\begin{bmatrix} M_\ell(F) & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & M_m(F) \end{bmatrix},$$

respectively.

(ii) Associated to a subalgebra A of $E_{(\ell,m)}(F)$ are homomorphic image subalgebras A_ℓ and A_m in $M_\ell(F)$ and $M_m(F)$ respectively.

(iii) Set

$$T_{(\ell,m)}(F) = \begin{bmatrix} 0 & M_{\ell\times m} \\ 0 & 0 \end{bmatrix},$$

the Jacobson radical of $E_{(\ell,m)}(F)$.

(iv) Recall that every F-algebra automorphism τ of $M_n(F)$ is *inner* (i.e., there exists an invertible Q in $M_n(F)$ such that $\tau(a) = QaQ^{-1}$ for all $a \in M_n(F)$). We will say that two F-subalgebras A, A' of $M_n(F)$ are *equivalent* provided there exists an automorphism τ of $M_n(F)$ such that $\tau(A) = A'$.

Lemma 2.1. *Let A be a subalgebra of $E_{(\ell,m)}(F)$ such that A_ℓ satisfies h_q and A_m satisfies h_r. Then A satisfies $h_{(q+r)}$.*

Proof. The hypothesis that A_ℓ satisfies h_q implies that the evaluation of h_q on A consists of matrices of the form

$$\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}.$$

Similarly, the hypothesis that A_m satisfies h_r implies that the evaluation of s_r on A consists of matrices of the form

$$\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$

Thus A satisfies $h_q\, h_r$. Since h_{q+r} is a linear combination of evaluations of $h_q\, h_r$, A satisfies h_{q+r}. $\square$

Theorem 2.2. *h_{4n-2} is an identity for any proper subalgebra of $M_n(F)$.*

Proof. Let A be a proper subalgebra of $M_n(F)$. If A is simple, then it is a a finite dimensional central simple algebra over its center k. Let K denote the algebraic closure of k; then $A \otimes_k K$ is a simple K-algebra in a natural way (cf. [11], §1.8), with $\dim_K (A \otimes_k K) = \dim_k(A)$. Also, $A \otimes_k K \cong M_t(K)$ for some $t \leq n$. Since A is a proper subalgebra of $M_n(F)$ it follows that $t < n$. Hence, by the Amitsur-Levitzki theorem, $A \otimes_k K$ satisfies s_{2n-2}. Since h_{4n-5} lies in the T-ideal generated by s_{2n-2}, we have that $h_{4n-5}(A) = 0$. If A is not simple, it can be embedded as F-algebra in $E_{(\ell,m)}(F)$ for some suitable positive integers ℓ and m (with $\ell + m = n$). Since $h_{4\ell-1}$ and h_{4m-1} are identities for $M_\ell(F)$ and $M_m(F)$ respectively, we apply Lemma 2.1 to obtain that h_{4n-2} is an identity for A. □

3. A Polynomial test for $E_{(\ell,m)}$

In this section we show that the double Capelli polynomial h_{4n-3} is a polynomial test for the subalgebra $E_{(\ell,m)}$ of $M_n(F)$ for any positive integers ℓ, m such that $\ell + m = n$.

Proposition 3.1. h_{4n-3} is an identity for every proper subalgebra of $E_{(\ell,m)}$.

Proof. We consider all possible proper subalgebras of $E_{(\ell,m)}(F)$. Let first consider a subalgebra A of $E_{(\ell,m)}$ such that A_ℓ is a proper subalgebra of $M_\ell(F)$. Then $h_{4\ell-2}$ is an identity for A_ℓ as established in Theorem 2.2, and h_{4m-1} is an identity for $M_m(F)$. Thus, by Lemma 2.1, h_{4n-3} is an identity for

$$\begin{bmatrix} A_\ell & M_{\ell\times m}(F) \\ 0 & M_m(F) \end{bmatrix},$$

and consequently an identity for A. Similarly, h_{4n-3} is an identity for every subalgebra of $E_{(\ell,m)}$ such that A_m is a proper subalgebra of $M_m(F)$. Clearly, h_{4n-4} is an identity for the semisimple case

$$\begin{bmatrix} M_\ell(F) & 0 \\ 0 & M_m(F) \end{bmatrix}.$$

It only remains to consider the case when the projections $A \to A_\ell$ and $A \to A_m$ are equivalent representations of A, which means that A there is a fixed matrix T such that $TA_\ell T^{-1} = A_m$. It easily follows that in this case A is equivalent to the F-subalgebra of the form

$$\left\{ \begin{bmatrix} a & c \\ 0 & a \end{bmatrix} : a, c \in M_\ell(F) \right\}.$$

In [2], Proposition 2.5, it is proved that the standard polynomial $s_{2\ell}$ is an identity for this algebra, hence, h_{2n-1} is an identity for A. □

Remark 3.2. The polynomial h_{4n-3} is not an identity for $E_{(\ell,m)}$. For instance, if $n = 3$ and $A = E_{(1,2)}$, we have

$$h_9 (e_{11}, e_{11}, e_{12}, e_{22}, e_{22}, e_{23}, e_{33}, e_{33}, e_{32}) = 2e_{12}.$$

Remark 3.3. The above ideas can be generalized to prove that the double Capelli polynomial h_{4n-t-1} is a polynomial test for the block upper triangular matrix

algebra
$$\begin{pmatrix} M_{\ell_1}(F) & & & * \\ & M_{\ell_2}(F) & & \\ & & \ddots & \\ & 0 & & M_{\ell_t}(F) \end{pmatrix}.$$

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